Dynamic Response of Rectangular Orthotropic Plates Based on Characteristic Orthogonal Polynomials and Rayleigh-Ritz Model

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Abstract: In this study, the frequencies and mode shapes of thin rectangular plates have been obtained using the Rayleigh Ritz method and characteristic orthogonal polynomials for plates of various boundary conditions. The maximum and minimum frequencies of 23.95 and 9.43 rad/sec were obtained for reinforced concrete plates with all round rigid fixity for aspect ratios of 0.5 and 1.0 respectively. These results clearly demonstrate the crucial role of aspect ratios and boundary restraints. Plates clamped on two adjacent edges and simply supported on the other two (CCSS) and plates clamped on two opposite edges and simply supported on the other two opposite edges (CSCS) yielded intermediate values of fundamental frequencies with the plate with similar restraints on contiguous sides showing the lesser values. It is noteworthy that plate clamped on opposite side and simply supported on the opposite side gave a maximum frequency almost equal to that for all round clamped plate. Rectangular plates with simply supported edges showed the least values of maximum and minimum of 12.08 and 5.17 rad/sec for aspect ratio of 0.5 and 1.0 respectively. The mode shapes followed a sequence of transverse vibrations along the longitudinal, transverse and diagonal axes of the plates. However, there were variations dependent on aspect ratio and boundary conditions. The output of the proposed model is in close agreement with those of other researchers and standard computer software.

Keywords: rayleigh-ritz, characteristic orthogonal polynomials, reinforced concrete, rectangular plates, vibration, ANSYS

1. INTRODUCTION

Plates are extensively used in several engineering fields such as civil, military, marine, and aerospace. In Civil Engineering, they are used in bridge decks, offshore platforms and are predominant in multi-storey buildings. Such wide application of plates across various engineering disciplines makes their vibrational behaviour an important consideration in their structural analysis and design. The transverse vibration of plates has been a subject of many excellent books including Timoshenko and Krieger [1] among others. A lot of research papers have also been published especially on the transverse vibration of rectangular isotropic plates much more than for orthotropic plates. However, the governing equations for a plate, represented by higher order differential equations, are rather difficult to solve analytically except for simple loadings and boundary conditions. This informed the need to evolve approximate methods which are easy to use and can be applied to plates with various configurations.

Approximate general solutions for the natural frequency of orthotropic plates have been achieved with Rayleigh-Ritz method in combination with characteristic beam functions by Hearmon [2], decomposition techniques by Bazely et al. [3] and Marangoni et al.[4] and Gorman superstition by Yu [5]. Further developments include the use of pseudo-Fourier expansion by Rossi et al. [6] and the finite difference method by Chen [7] and Huang et al. [8]. The finite element method has also been used to study dynamic response of complicated cases of laminated skewed sandwich plates and tapered beams in the works of Chalak et al. [9] and Bayat et al. [10] respectively.

Recently, Osadebe et al. [11] investigated the dynamic response of orthotropic plates with the use of characteristic orthogonal polynomials in combination with Galerkin variational principle. However, the models obtained were limited to the determination of the fundamental natural frequencies of vibration only. It is obvious that the study of dynamic response without the analysis of the displacement profile and mode shape is incomplete. In order to address this limitation, the present study adopted the characteristic orthogonal polynomials in combination with Rayleigh-Ritz principle to derive new analytical models for frequency and mode shapes of orthotropic rectangular plates with mixed clamped and simply supported edges.

2. PROBLEM FORMULATION

The problem formulation involved

a. derivation of equations for natural frequency of orthotropic rectangular plates based on Rayleigh-Ritz principle.

b. development of characteristic orthogonal polynomials for rectangular plates with various combinations of clamped and simply supported boundary conditions.

c. determination of frequencies and mode shapes.

2.1 Formulation of Rayleigh-Ritz Equation for Natural Frequency of Orthotropic Rectangular Plates

The Rayleigh-Ritz principle states that, provided the vibrating system is conservative, the maximum kinetic energy, T_{max}, must be equal to the maximum potential or strain energy, U_{max}. For a plate of mass density ρ and constant thickness h, the kinetic energy in two dimensions is represented by the expression

$$T = \frac{1}{2}\int_A \rho h \left[\frac{\partial w(x,y,t)}{\partial t}\right]^2 dA \tag{1}$$

where - w(x,y) is the transverse displacement of a point located at (x,y) and is usually assumed in the form of a continuous function that approximately represents the shape of the plate's deflected middle surface.

Assuming that the plate is undergoing harmonic vibrations, then the vibrating middle surface of the plate can be approximated by

$$w(x,y,t) \equiv w(x,y) \sin \omega t \tag{2}$$

where ω represents the natural frequency of the plate pertinent to the assumed shape function.

Substituting (2) into (1) we have

$$T = \frac{\omega^2}{2}\cos^2\omega t \int_A \rho h w^2(x,y) \, dA \tag{3}$$

But the kinetic energy is maximum when $\cos^2\omega t = 1$

Therefore,

$$T_{max} = \frac{\omega^2}{2}\int_A \rho h w^2(x,y) \, dA \tag{4}$$

The maximum potential energy for orthotropic plate is expressed by

$$U_{max} = \frac{1}{2}\int_A \left[D_x\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_{xy}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right) + D_y\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4D_s\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right] dA \tag{5}$$

Equations (4) and (5) can be represented in non-dimensional form using the following coordinate redefinition x= aR and y = bQ

where R and Q are coordinates in the X and Y directions respectively

$$\text{Then,} \quad T_{max} = \frac{\omega^2}{2}\int_0^1\int_0^1 \rho h w^2(R,Q) \, dRdQ \tag{6}$$

and

$$U_{max} = \frac{1}{2} \int_0^1 \int_0^1 \left[\frac{D_x}{a^4} \left(\frac{\partial^2 w}{\partial R^2} \right)^2 + 2 \frac{D_{xy}}{a^2 b^2} \left(\frac{\partial^2 w}{\partial R^2} \frac{\partial^2 w}{\partial Q^2} \right) + \frac{D_y}{b^4} \left(\frac{\partial^2 w}{\partial Q^2} \right)^2 + \frac{4D_s}{a^2 b^2} \left(\frac{\partial^2 w}{\partial R \, \partial Q} \right)^2 \right] dRdQ \qquad (7)$$

Equating maximum kinetic energy to maximum total strain energy and introducing the aspect ratio r = b/a, the expression for natural frequency of vibration of orthotropic plate has the form

$$\omega^2 = \frac{\int_0^1 \int_0^1 \left[D_x \left(\frac{\partial^2 w}{\partial R^2} \right)^2 + \frac{2D_{xy}}{r^2} \left(\frac{\partial^2 w}{\partial R^2} \frac{\partial^2 w}{\partial Q^2} \right) + \frac{D_y}{r^4} \left(\frac{\partial^2 w}{\partial Q^2} \right)^2 + \frac{4D_s}{r^2} \left(\frac{\partial^2 w}{\partial R \, \partial Q} \right)^2 \right] dRdQ}{\rho h a^4 \int_0^1 \int_0^1 w^2 \, dRdQ} \qquad (8)$$

where

$$D_x = \frac{E_x h^3}{12(1 - \mu_x \mu_y)}; \quad D_y = \frac{E_y h^3}{12(1 - \mu_x \mu_y)}; \quad D_{xy} = \frac{E_x \mu_y h^3}{12(1 - \mu_x \mu_y)}; \quad D_s = \frac{Gh^3}{12}; \frac{\mu_x}{E_x} = \frac{\mu_y}{E_y}; \text{ r = b/a}$$

In the above equations, *A* is the area of the plate, D_x, D_y, D_{xy}, and D_s are the flexural and torsional rigidities of the plate, ρ is the mass density of the plate, h is the plate thickness, G, E_x and E_y represent the shear and elastic moduli of elasticity of the plate material respectively.

2.2 Development of Characteristic Orthogonal Polynomials (COPs)

Characteristic orthogonal functions were derived for four typical cases of rectangular plates defined by their boundary conditions.

Case 1: Rectangular plate simply supported and clamped on opposite sides (CSCS)

Figure 1 shows a sketch of the plate and its edge restraints, defined in the non dimensional
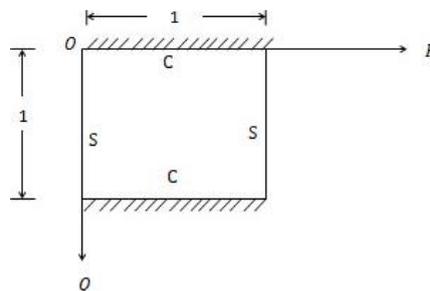
coordinate system R, Q.

Figure 1: Thin Rectangular Plate with CSCS Edge Restraints

The transverse displacement of the plate can be approximated by the product of two orthogonal polynomial functions in R and Q as follows

$$w(R, Q) = (A_0 + A_1 R + A_2 R^2 + A_3 R^3 + A_4 R^4)(B_0 + B_1 Q + B_2 Q^2 + B_3 Q^3 + B_4 Q^4) \quad (9)$$

The boundary conditions are as follows: deflections at all edges are zero, slope at edge

Q = 0 or 1 is zero, and moment at edge R = 0 or 1 is zero.

Substituting these boundary conditions into (9), the shape function for plate clamped and simply supported on two opposite edges (CSCS) becomes

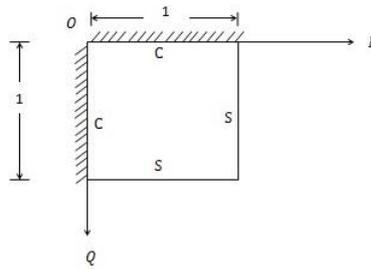$$w(R, Q) = K(R - 2R^3 + R^4)(Q^2 - 2Q^3 + Q^4) \qquad (10)$$

where $K = A_4 B_4$

Substituting (10) into (8), the fundamental frequency expression for the plate has the form

$$\omega^2 = \frac{\pi^2}{a^2 r^2} \sqrt{\frac{1}{\rho h} \left(1.0014 D_x \, r^4 + 2.4316 H r^2 + 5.1741 D_y \right)} \qquad (11)$$

Case 2: Rectangular plate clamped on adjacent edges and simply supported on others (CCSS)

The plate is shown in Figure 2. The following boundary conditions apply

Figure 2: Thin Rectangular Plate with CCSS Edge Restraints

Deflection along all edges are zero, slope at edges R = 0 or Q = 0 is zero, and moment at edges R =1 or Q = 1 is zero. Repeating operations similar to those for case 1, the shape function can be expressed as

$$w(R,Q) = K(1.5R^2 - 2.5R^3 + R^4)(1.5Q^2 - 2.5Q^3 + Q^4) \qquad (12)$$

For plate clamped and simply supported on two adjacent edges, the fundamental frequency expression is

$$\omega^2 = \frac{\pi^2}{a^2 r^2} \sqrt{\frac{1}{\rho h} \left(2.4508 D_x \, r^4 + \, 2.6547 H r^2 + 2.40508 D_y\right)} \qquad (13)$$

Case 3: **Rectangular plate simply supported on all edges (SSSS)**

Figure 3 illustrates a sketch of the plate. The following boundary conditions apply: deflections at all edges are zero, moment at all edges is zero.
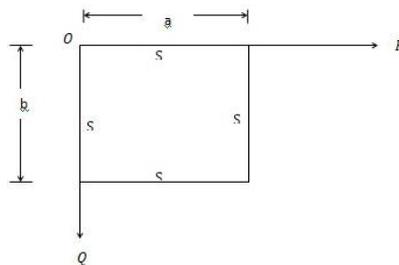
Figure 3: Thin Rectangular Plate Simply Supported on all edges (SSSS)

Performing similar mathematical substitutions as in previous cases, the shape function becomes

$$w(R,Q) = K(R - 2R^3 + R^4)(Q - 2Q^3 + Q^4) \qquad (14)$$

The fundamental frequency expression of this plate is given as

$$\omega^2 = \frac{\pi^2}{a^2 r^2} \sqrt{\frac{1}{\rho h} \left(D_x \, r^4 + \, 2H r^2 + D_y \right)} \qquad (15)$$

Case 4: Rectangular plate clamped on all four edges (CCCC)

The plate is shown in Figure 4 and has the following boundary conditions: deflections, slopes and moments at R = 0, 1; and Q = 0, 1 are zero.
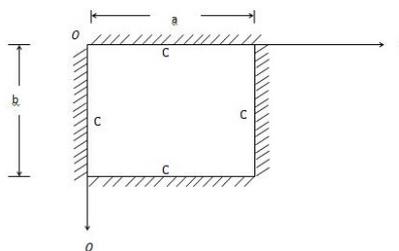
Figure 4:Thin rectangular plate clamped on all edges (CCCC)

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Repeating the usual operations as in previous cases, the deflection function is expressed as
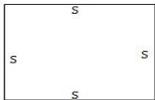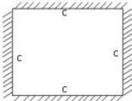
$$w(R,Q) = K(R^2 - 2R^3 + R^4)(Q^2 - 2Q^3 + Q^4) \qquad (16)$$

The fundamental frequency expression of this plate is given as

$$\omega^2 = \frac{16.9703}{a^2 r^2} \sqrt{\frac{1}{\rho h}\left(1.75D_x \, r^4 + H\, r^2 + 1.75D_y \right)} \qquad (17)$$

The expressions for the characteristic orthogonal shape functions and fundamental natural frequencies of rectangular orthotropic plates of various boundaries conditions considered are summarized in Table 1.

<div align="center">

TABLE 1: SHAPE FUNCTION AND FREQUENCY EXPRESSION OF VARIOUS PLATES

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Plate	Shape Function and Frequency Expression
CSCS	$w(R,Q) = k(R - 2R^3 + R^4)(Q^2 - 2Q^3 + Q^4)$ $\omega^2 = \frac{\pi^2}{a^2 r^2}\sqrt{\frac{1}{\rho h}\left(1.0014D_x \, r^4 + 2.4316Hr^2 + 5.1741D_y\right)}$
CCSS	$w(R,Q) = k(1.5R^2 - 2.5R^3 + R^4)(1.5Q^2 - 2.5Q^3 + Q^4)$ $\omega^2 = \frac{\pi^2}{a^2 r^2}\sqrt{\frac{1}{\rho h}\left(2.4508D_x \, r^4 + 2.6547Hr^2 + 2.40508D_y\right)}$
SSSS	$w(R,Q) = k(R - 2R^3 + R^4)(Q - 2Q^3 + Q^4)$ $\omega^2 = \frac{\pi^2}{a^2 r^2}\sqrt{\frac{1}{\rho h}\left(D_x \, r^4 + 2Hr^2 + D_y\right)}$
CCCC	$w(R,Q) = K(R^2 - 2R^3 + R^4)(Q^2 - 2Q^3 + Q^4)$ $\omega^2 = \frac{16.9703}{a^2 r^2}\sqrt{\frac{1}{\rho h}\left(1.75D_x \, r^4 + H\, r^2 + 1.75D_y\right)}$
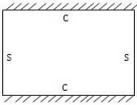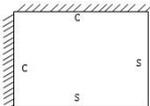
2.3 Solution Techniques for Frequencies and Mode Shapes for Reinforced Concrete Slab in Rayleigh- Ritz Method

The following procedure was developed for the solution of the equations in Table 1, applicable to reinforced concrete slabs treated in Osadebe et al. [11].

2.3.1 Determination of Fundamental Frequencies

1. Calculate moment of inertia, for steel about an axis, in the xx-axis and yy-axis using $I_{x'x'} = S_{vx}\frac{\pi \emptyset_{sx}^4}{64}$ and $I_{y'y'} = S_{vy}\frac{\pi \emptyset_{sy}^4}{64}$ respectively.

2. Evaluate area of steel, $= S_{vx}\,\pi\frac{\emptyset^2}{4}$, $S_{vy}\,\pi\frac{\emptyset^2}{4}$

3. Calculate effective depth, d

4. Evaluate moment of inertia of steel bars about x and y directions $I_{xs} = Ix'x' \; Ad^2$ and $I_{ys} = Iy'y' + Ad^2$

5. Calculate flexural rigidities about x- and y – axes, D_x and D_y

6. Evaluate $H = \sqrt{D_x D_y}$

7. Calculate ω^2 from relevant equation in Table 1.

2.3.2 Determination of Other Frequencies and Mode Shapes

Modal analyses of rectangular reinforced concrete slabs of aspect ratios and boundary conditions were performed using ANSYS software. It is known that, there are many types of elements in this software. Some of the elements are suitable for modelling reinforced concrete slabs. SHELL 181 was adopted for this study. The slabs were split into 18×9 elements along the length and width with CCSS, CSCS, CCCC and SSSS boundary conditions.

The mode shapes provide useful information that can assist in the design of plates. The understanding and visualization of mode shapes is invaluable in the design process. They help to identify areas of weakness in the design or areas where improvement is needed. The development of a modal model is useful for simulation and design studies

3.   RESULTS AND DISCUSSION

The models have been derived on the basis of characteristic orthogonal polynomials (COP) and Rayleigh Ritz method. The validation of the accuracy and reliability of the model was carried out by means of numerical examples and computation with the use of ANSYS computer software. Four cases, involving the combinations of clamped and simply supported edge conditions were considered.

3.1 Validation of Proposed Model

For purposes of validation of the proposed model, the problems treated in Osadebe et al. [11] using the Galerkin method were considered. The model parameters are as follows

Modulus of elasticity of steel, $E_s = 205,000$ N/mm^2

Concrete characteristics strength, $f_{cu} = 35$MPa

Possion's Ratio, $\mu_c = 0.20$

Thickness of slab, h = 175 mm,

Density of reinforced concrete, $\rho = 2563$ kg/m^3

a = 5000mm, for b/a = 0.5, 0.6, 0.7, 0.8, 0.9 and 1.0;

Cover to all reinforcement, c = 25 mm

Diameter of mild reinforcement steel bars = 12 mm

Spacing of steel reinforcement = 150 mm

The computation of frequencies was carried out using the procedure set out in section 2.3.1.The results of the computations using the derived equations are presented in Table 2, from where , it can be seen that the fundamental frequency of vibration of all four types of rectangular plates considered decreased with increase in aspect ratio.

Rectangular plates with simply supported edges showed the least values of maximum and minimum frequencies of 12.08 and 5.17 rad/sec for aspect ratio of 0.5 and 1.0 respectively. The greatest values of maximum and minimum frequencies of 23.95 and 9.43 rad/sec were obtained for plates with all round rigid fixity and aspect ratios of 0.5 and 1.0 respectively. These results clearly demonstrate the crucial role of aspect ratios and boundary restraints on vibration of plates. Plates with CCSS and CSCS edge restraints yielded intermediate values of fundamental frequencies with the plate with similar restraints on contiguous sides (CCSS) showing the lesser values. It is interesting to note that plate clamped and simply supported on the opposite sides (CSCS) yielded frequency values almost equal to those for all round clamped plate (CCCC).

The table further shows that the output from the proposed model are much closer to those obtained from ANSYS computer software, while the model by Osadebe et al. [11], based on Galerkin method, slightly underestimates the fundamental natural frequency of the plates under consideration. On the whole, the proposed model yielded results with about 99 percent accuracy compared to results from ANSYS computer software.

TABLE 2: VALIDATION OF FUNDAMENTAL NATURAL FREQUENCIES (ω) OF RECTANGULAR ORTHOTROPIC PLATES BASED ON THE PROPOSED MODEL

Plate Type	Aspect Ratio, r	Fundamental Frequency (ω) radians/sec based on various models		
		Osadebe et al. (2017)	Present Study (NUMERICAL)	Present Study (ANSYS)
SSSS	0.5	12.089119	12.082214	11.922972
	0.6	9.299012	9.262356	9.196698
	0.7	7.600141	7.596269	7.539822
	0.8	6.507281	6.504073	6.471053
	0.9	5.732241	5.729466	5.726998
	1.0	5.172742	5.170251	5.178978
CCSS	0.5	17.394250	17.276528	17.179485
	0.6	13.141580	13.034195	13.036352
	0.7	10.596813	10.543409	10.548839
	0.8	8.994340	8.956029	8.974901
	0.9	7.885202	7.857215	7.859008
	1.0	7.105523	7.084651	7.142096
CSCS	0.5	23.088498	23.088122	22.951847
	0.6	16.785503	16.763915	16.819458
	0.7	12.957086	12.966065	13.051432
	0.8	10.525836	10.525628	10.627379
	0.9	8.809916	8.809724	8.913954
	1.0	6.272057	7.584096	7.684335
CCCC	0.5	23.956242	23.954457	23.749812
	0.6	17.854125	17.828493	17.832937
	0.7	14.241764	14.240773	14.293618
	0.8	12.000379	11.999563	12.078167
	0.9	10.477434	10.476731	10.611672
	1.0	9.429892	9.429261	9.572433

The closeness of these results is evident from the plots of Figures 5 – 8, showing the variation of the fundamental frequencies of the various plates analysed.
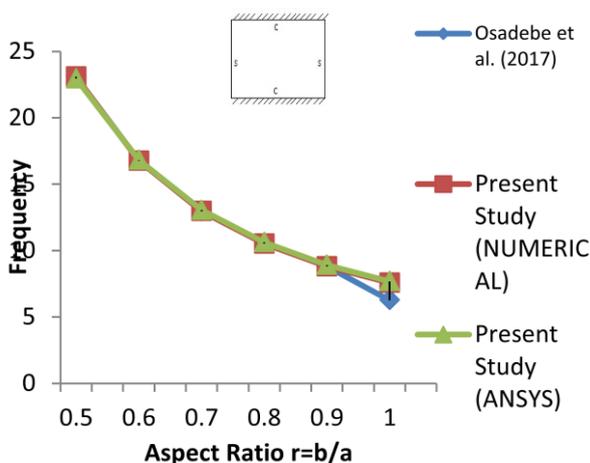
Figure 5: Fundamental natural frequencies, ω of plate simply supported on two opposite edges and clamped on the other two (CSCS)
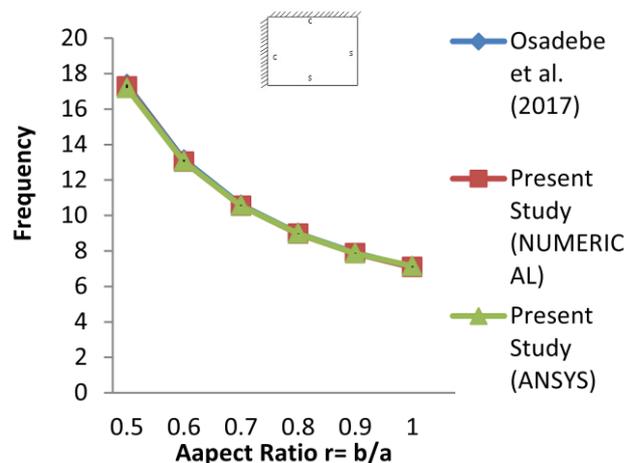
Figure 6: Fundamental natural frequencies, ω of plates clamped on the adjacent edges and simply supported on the two adjacent far edges (CCSS)
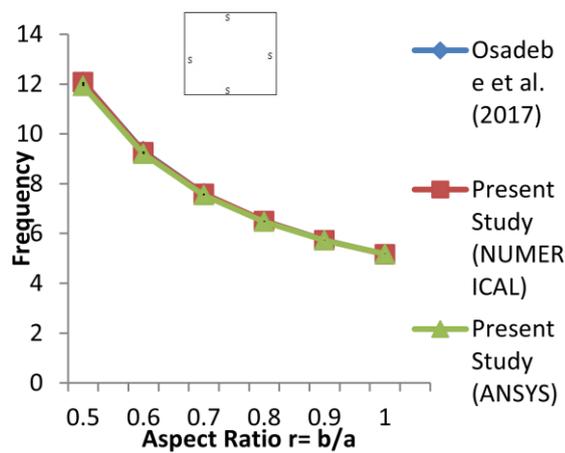
Figure 7: **Fundamental natural frequencies, ω of plates simply supported on all edges (SSSS)**
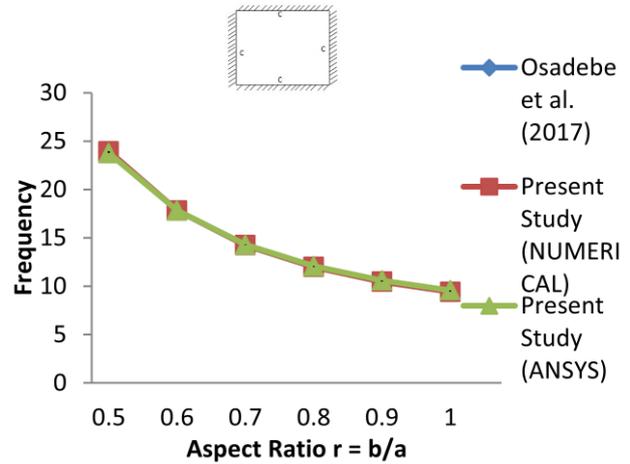
Figure 8: **Fundamental natural frequencies, ω of plates clamped on all edges (CCCC)**

3.2 Prediction of Higher Frequencies and Mode Shapes

The computed higher frequencies are shown in Table 3 for the four types of plates considered.

TABLE 3: PREDICTED FIRST FIVE FREQUENCIES IN RAD/SEC

Plate Type	Aspect Ratio	1st Frequency	2nd Frequency	3rd Frequency	4th Frequency	5th Frequency
SSSS	0.5	12.082214	19.347184	32.121528	41.242828	47.991597
	0.6	9.262356	16.805007	29.985873	30.126616	37.165669
	0.7	7.596269	15.338511	23.264750	29.127590	30.560156
	0.8	6.504073	14.324405	18.822538	26.256174	28.177572
	0.9	5.729466	13.873273	15.650786	23.386644	29.018891
	1.0	5.170251	13.354282	13.354910	21.177476	28.540740
CCSS	0.5	17.276528	24.623803	37.843625	51.513323	56.987234
	0.6	13.034195	21.013484	35.046979	36.660784	44.572916
	0.7	10.543409	19.026741	29.052192	34.057377	36.431165
	0.8	8.956029	17.702874	23.469582	31.168369	32.933315
	0.9	7.857215	17.227865	19.482272	27.736493	34.333837
	1.0	7.084651	16.567503	16.641016	25.109493	33.785944
CSCS	0.5	23.088122	27.940697	38.307952	54.907499	62.567959
	0.6	16.763915	22.293998	33.691696	45.664934	50.878093
	0.7	12.966607	19.004122	31.463050	35.024988	40.549164
	0.8	10.525628	16.915591	28.056307	29.783554	33.831811
	0.9	8.809724	15.711733	23.022219	29.147068	30.091431
	1.0	7.584096	14.728414	19.369803	25.710794	29.333050
CCCC	0.5	23.954457	30.991183	44.362429	62.976366	64.075923
	0.6	17.828493	25.974688	40.679854	46.177642	52.923269
	0.7	14.240773	23.306219	35.648908	39.513695	43.061810
	0.8	11.999563	12.078167	21.580856	28.792068	38.169094
	0.9	10.476731	21.105847	23.887413	32.665023	40.332394
	1.0	9.429261	20.354378	20.355007	29.568041	39.659465
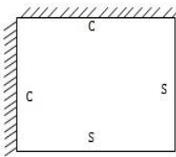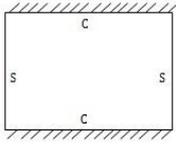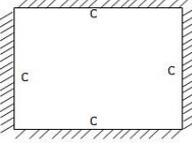
From Table 3, it can be seen that the frequencies of vibration of all four types of rectangular plates considered decreased with increase in aspect ratio. For plate with CSCS edge restraints, the frequency dropped by about three times from 23.1 rad/sec as aspect ratio increased from 0.5 to 1.0. A similar trend was observed for 2nd – 5th frequency, but with lesser rate of decrease.

The variations of first five frequencies with aspect ratio for plates simply supported on two opposite edges and clamped on the other two (CSCS) and plates clamped on the adjacent edges and simply supported on the two adjacent far edges (CCSS) are shown in Figures 9 and 10.
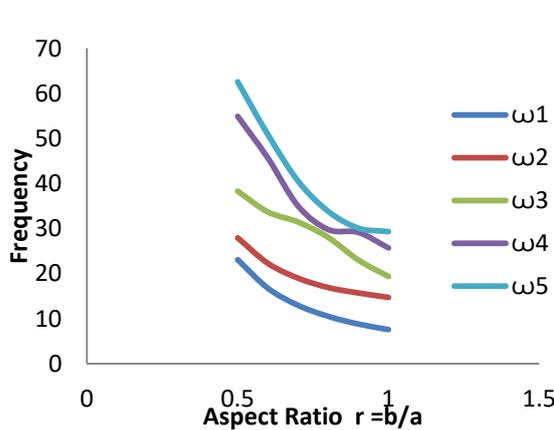
Figure 9: **Variation of first five frequencies with aspect ratio for CSCS plate**
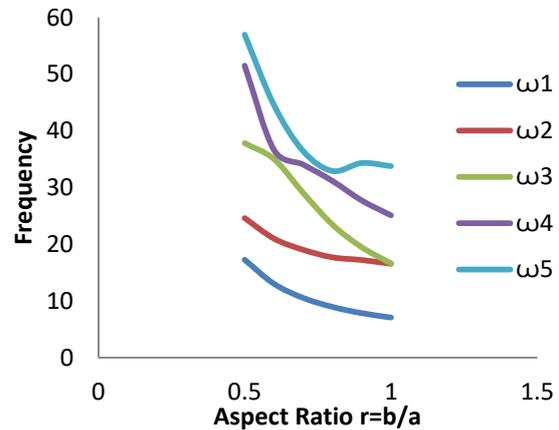
Figure 10: **Variation of first five frequencies with aspect ratio for CCSS plate**

For plate with CSCS edge restraint, all five frequencies ω_1, ω_2, ω_3, ω_4 and ω_5, exhibit a downward trend between aspect ratios of 0.5 and 1. For plates clamped and simply supported on two adjacent edges (CCSS), the first four frequency values decreased gradually with increase in aspect ratio. The fifth frequency value, however, exhibited an upward turning point between aspect ratios 0.8 and 1.

Figures 11 and 12 show the variation of first five frequencies with aspect ratio for plates simply supported on all edges (SSSS) and plates clamped on all four edges (CCCC).
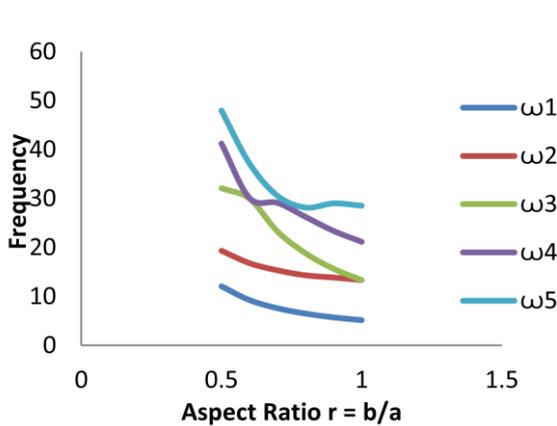
Figure 11: **Variation of first five frequencies with aspect ratio for SSSS plate**
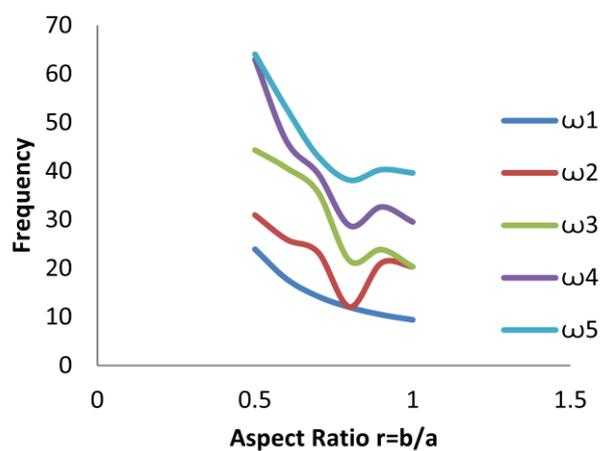
Figure 12: **Variation of first five frequencies with aspect ratio for CCCC plate**

For all round simply supported plate, four frequencies ω_1, ω_2, ω_3, and ω_5, exhibit a downward trend between aspect ratios 0.5 and 1.However, the fourth frequency ω_4 appeared to maintain a constant value within the range of aspect ratio considered. From Figure 12, it can be seen that the plate clamped on all edges also showed a sharp downward peak at an aspect ratio of about 0.8. The maximum frequency value for plates clamped on all edges was 40.332394 rad/sec and the minimum was 9.429261 rad/sec. For plates simply supported on all edges, the maximum frequency was 37.165669 rad/sec and the least frequency was 5.170251 rad/sec. The maximum and minimum frequencies for plates with mixed support fell between these limits.

3.3 Analysis of Mode Shapes for Plates

Modal analyses of rectangular reinforced concrete slabs of aspect ratios 0.5, 0.6, 0.7, 0.8, 0.9, and 1.0 were performed with a combination of clamped and simply supported boundary conditions using ANSYS software. As mentioned earlier, the slabs were split into 18×9 elements along the length and width of the plates. The output of the computations for all round simply supported plate with aspect ratio of 0.5 is shown in the pictorial representation of the mode shapes in Figure 13-16. From these figures, the following observations on the trend of mode shapes can be made:

From Figures 13, 14, 15 and 16, it can be seen that plates with aspect ratio of 0.5 exhibited different mode shapes depending on their edge restraints. The plate with SSSS edge restraints with aspect ratio of 0.5 exhibited waveline mode shapes in the sequence of transverse vibration along the centroidal longitudinal axis for the first three frequencies, followed by transverse vibration along the centroidal transverse axis for the fourth frequency and then a torsional vibration in two centroidal diagonal axes. The first frequency exhibited one peak at the centre of plate, the second and third had two and three peaks symmetrically located about the centre. The fourth frequencies were associated with two peaks in the transverse axis equidistant from the centre while the fifth frequency had a peak frequency in the centre of each quadrant of the plate

The plate with rigid restraints on all edges (Figure 14) exhibited a similar trend in mode shape variation for the first four frequencies. However, the fifth frequency yielded a transverse vibration with four peaks in the longitudinal axis. There was no torsional vibration.

The mode shapes for the plate with CSCS edge restraint (Figure 15) were characterized by transverse vibration along the longitudinal axis for the first four frequencies followed by a transverse vibration in the centroidal transverse axis for the fifth frequency.

The plate with CCSS edge restraint (Fig. 16) had mode shapes similar to those of the rigidly fixed plate with CCCC edge conditions. The similarity between the behaviour of the plate with CCCC and CCSS restraints was observed earlier as their maximum and minimum frequencies were close in magnitude.

The mode shapes for plates with aspect ratio of 0.6 are presented in Appendix from where it can be observed that all the plates exhibited similar mode shapes of wavelike variation with a sequence of transverse vibration along the centroidal longitudinal axis for the first three frequencies followed by transverse vibration along the centroidal transverse axis and a torsional vibration along the diagonal axes. It follows that the mode shapes of the plates with higher aspect ratios tend towards those of an all round simply supported boundary restraint with aspect ratio of 0.5.
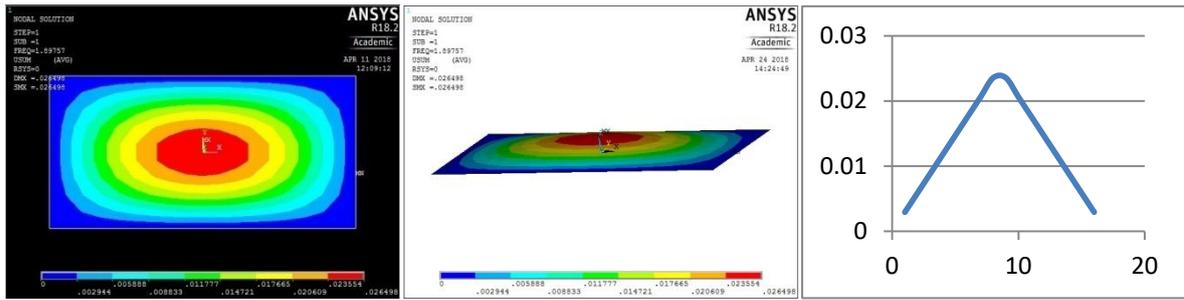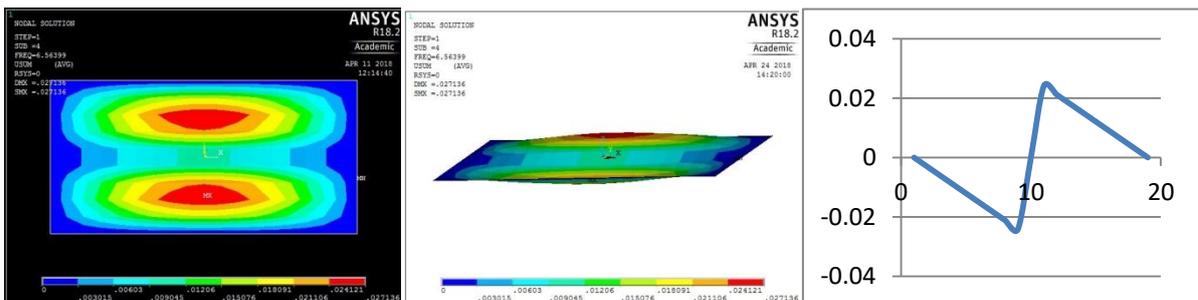
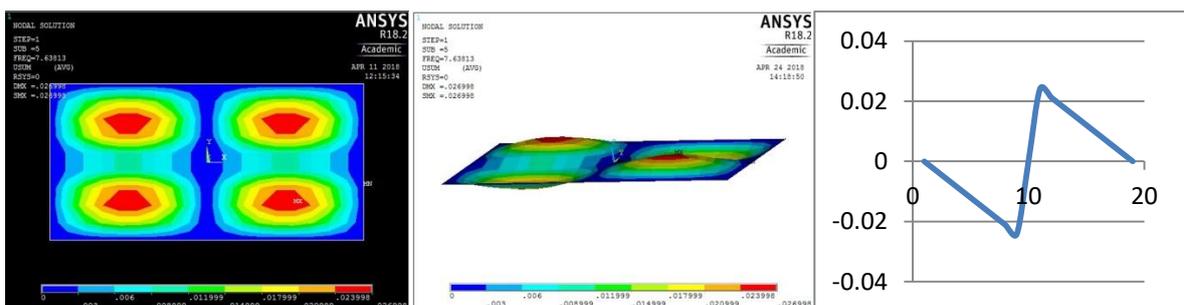1st vibration mode ω_1 = 11.922972 rad/sec
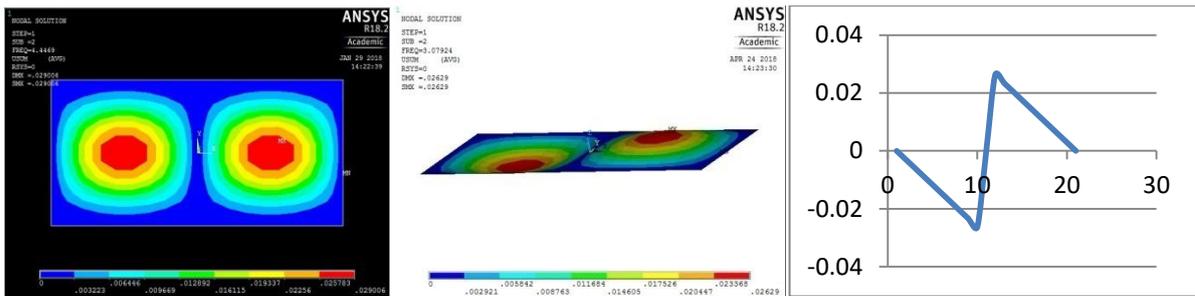
2nd vibration mode ω_2 = 19.347184 rad/sec
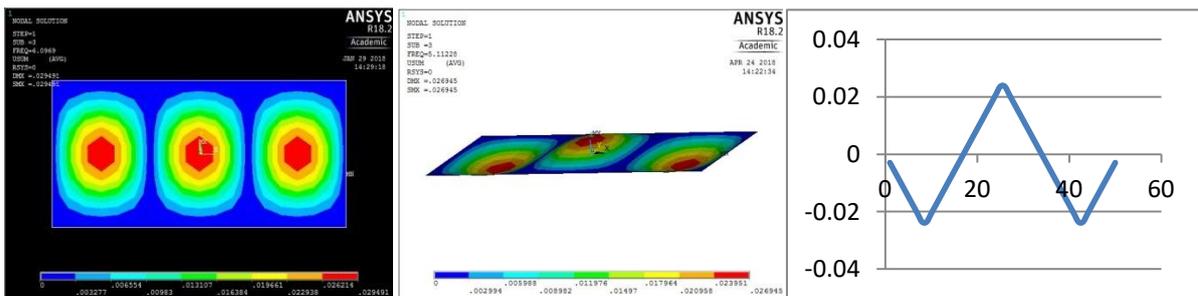
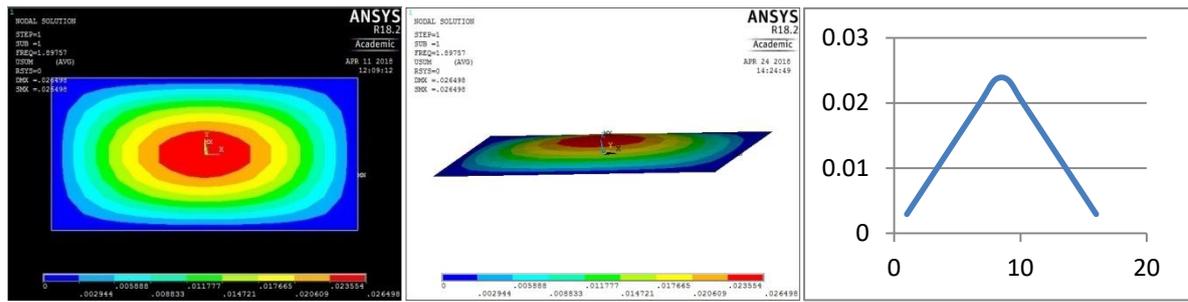3rd vibration mode ω_3 = 32.121528 rad/sec
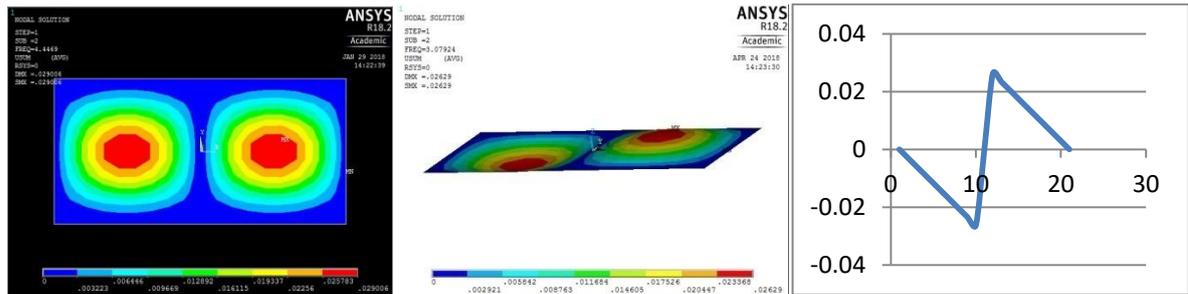

4th vibration mode ω_4 = 41.242828 rad/sec

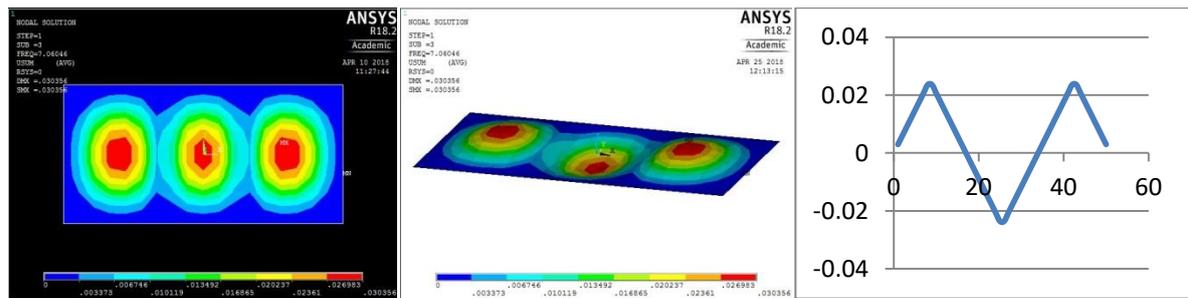
5th vibration mode ω_5 = 47.991598 rad/sec

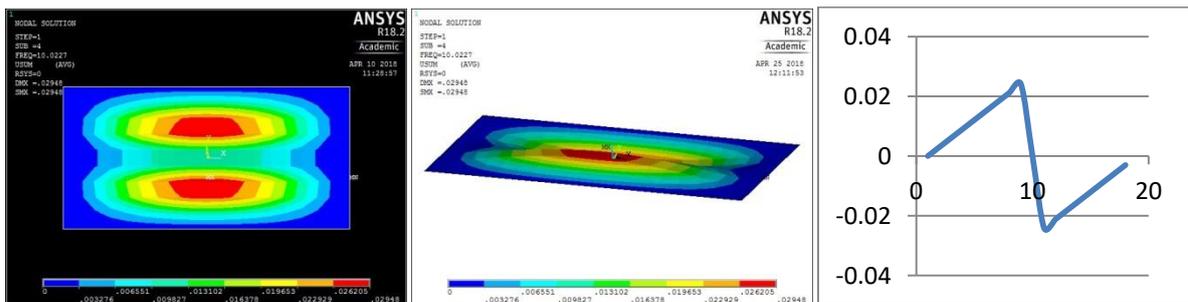**Figure 13: The first five mode shapes of SSSS plate (aspect ratio = 0.5)**
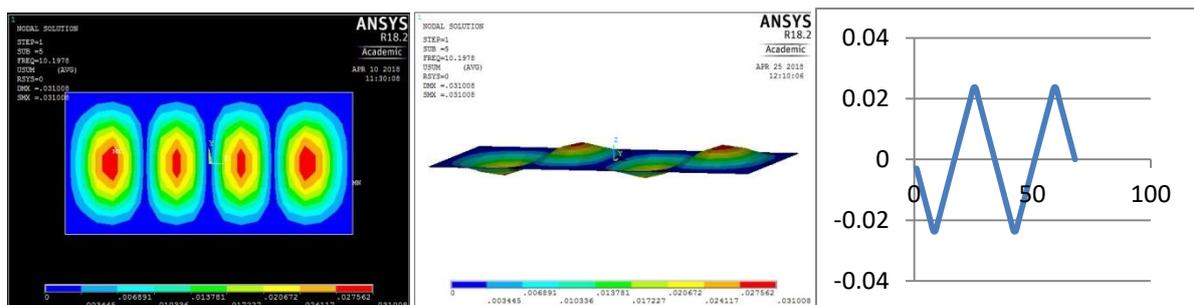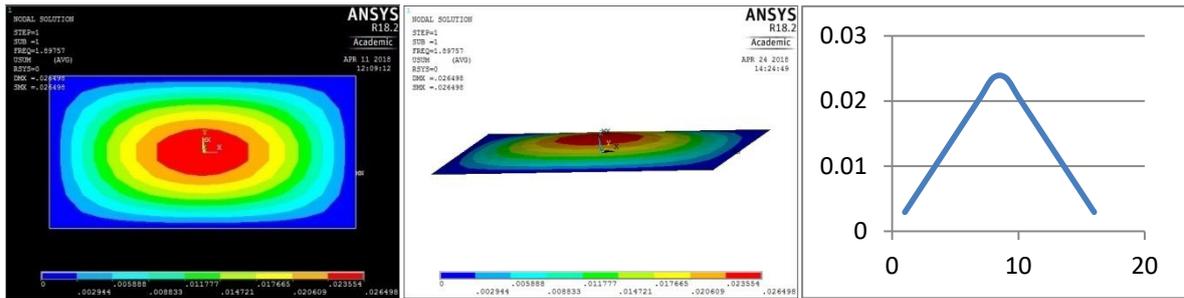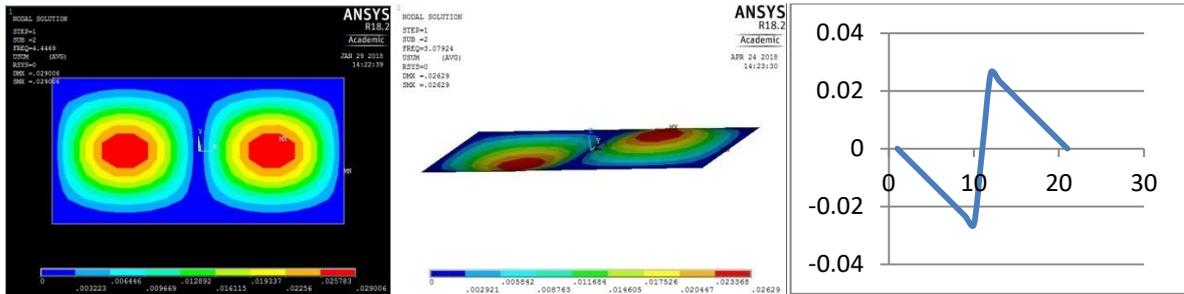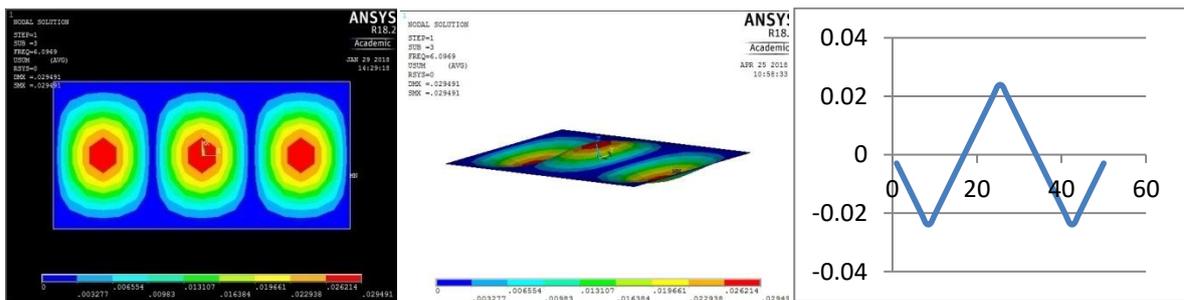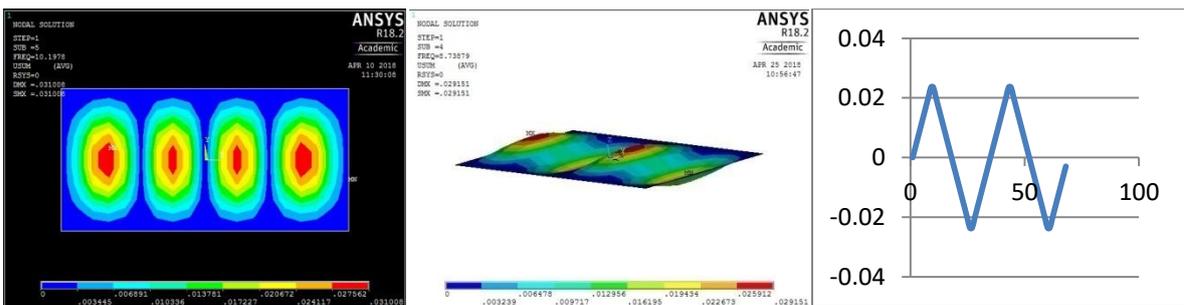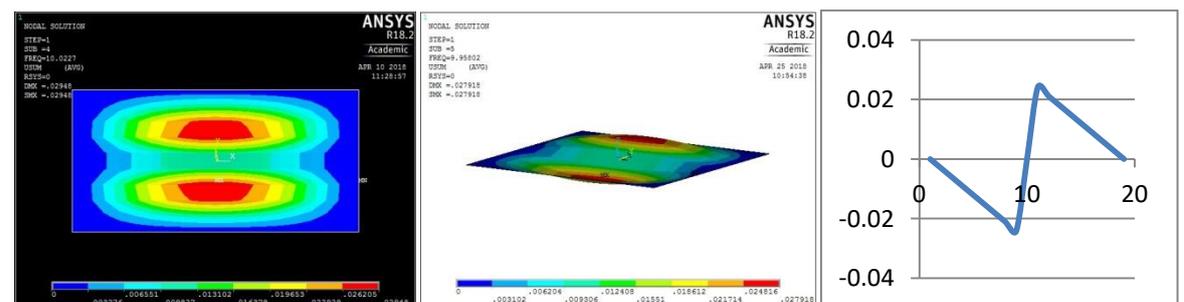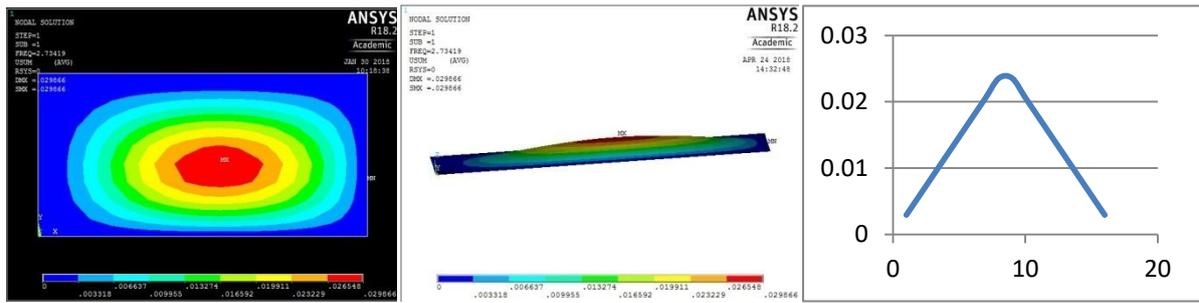
1st vibration mode $\omega_1 = 23.749812$ rad/sec

2nd vibration mode $\omega_2 = 30.991183$ rad/sec

3rd vibration mode $\omega_3 = 44.362430$ rad/sec

4th vibration mode $\omega_4 = 62.976366$ rad/sec

5th vibration mode $\omega_5 = 64.075924$ rad/sec

Figure 14: The first five mode shapes of CCCC plate (aspect ratio = 0.5)

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1st vibration mode ω_1 = 22.951848 rad/sec

2nd vibration mode ω_2 = 27.940697 rad/sec

3rd vibration mode ω_3 = 38.307952 rad/sec

4th vibration mode ω_4 = 54.907500 rad/sec

5th vibration mode ω_5 = 62.567959 rad/sec

Figure 15: The first five mode shapes of CSCS plate (aspect ratio = 0.5)

1st vibration mode ω_1 = 17.179485 rad/sec
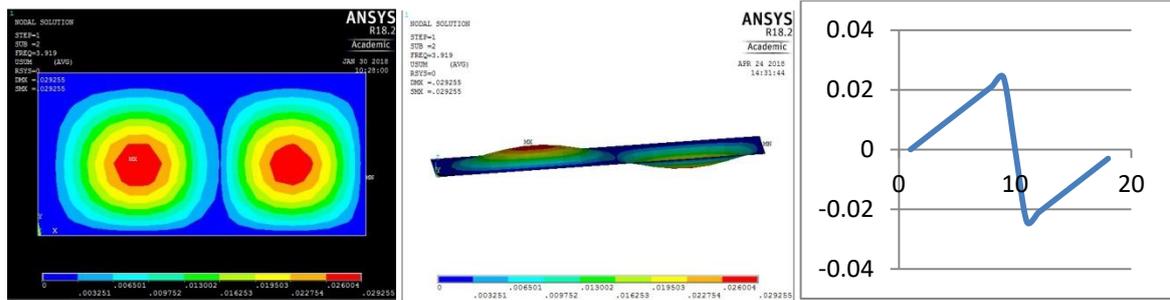
2nd vibration mode ω_2 = 24.623803 rad/sec
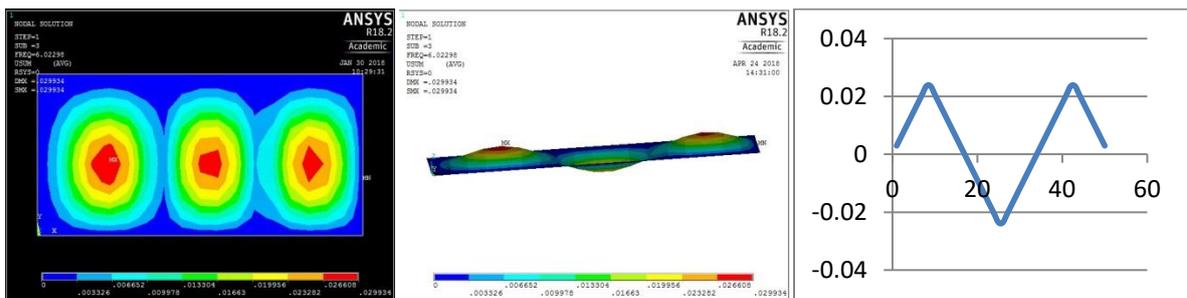
3rd vibration mode ω_3 = 37.843625 rad/sec
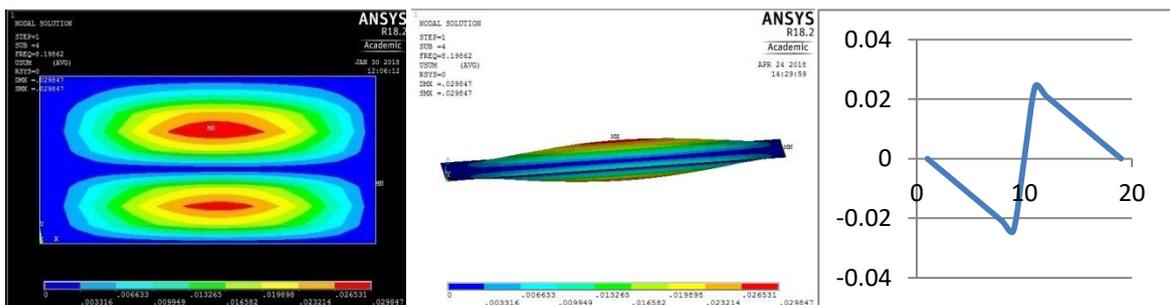
4th vibration mode ω_4 = 51.513323 rad/sec
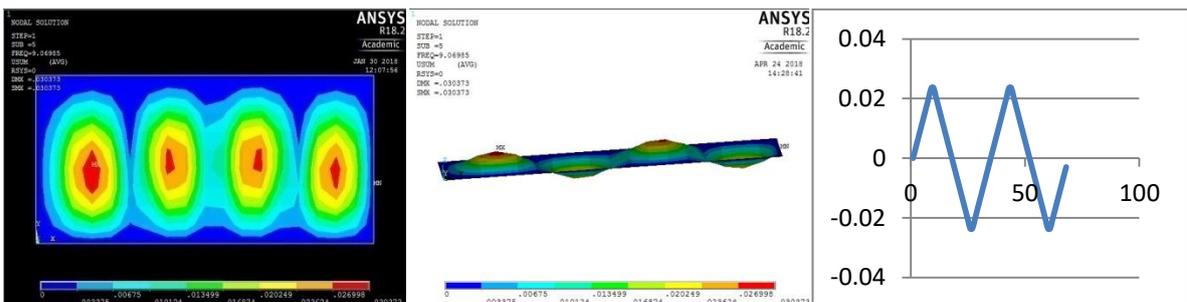
5th vibration mode ω_5 = 56.987234 rad/sec

Figure 16: The first five mode shapes of CCSS plate (aspect ratio = 0.5)

4. CONCLUSION AND RECOMMENDATIONS

4.1 Conclusion

In this work, the characteristic orthogonal polynomials have been applied to the Rayleigh-Ritz method for plates with four different boundary configurations. These include plates clamped on two adjacent edges and simply supported on the other two (CCSS), plates clamped on two opposite edges and simply supported on the other two opposite edges (CSCS), plate simply supported on all edges (SSSS), and plates clamped on all edges (CCCC). Based on the results obtained, it can be concluded that the model developed present an efficient technique for the evaluation and prediction of the fundamental frequencies and mode shapes of orthotropic plates with varying boundary conditions and aspect ratios.

Specifically, the maximum and minimum frequencies of 23.95 and 9.43 rad/sec were obtained for plates with all round rigid fixity for aspect ratios of 0.5 and 1.0 respectively. Plates with CCSS and CSCS edge restraints yielded intermediate values of fundamental frequencies with the plate with similar restraints on contiguous sides showing the lesser values. It is noteworthy that plate clamped on opposite side and simply supported on the opposite side gave a maximum frequency almost equal to that for all round clamped plate. Rectangular plates with simply supported edges showed the least values of maximum and minimum of 12.08 and 5.17 rad/sec for aspect ratio of 0.5 and 1.0 respectively. The model output was in close agreement with those of other researchers and standard computer software to about 99 percent accuracy.

The mode shapes for plates with aspect ratio of 0.5 significantly depend on the boundary conditions of the plate. For all-round simply supported plate, the mode shape represented a sequence of transverse vibration along the longitudinal axis for the first three frequencies, followed by transverse vibration along the transverse axis for the fourth frequency and torsional vibration for the fifth. Mode shapes for plates with other boundary conditions were characterised by slight changes in the sequence but without the torsional mode. The mode shapes for plates with aspect ratio of 0.6 tend to follow the trend for simply supported plate irrespective of boundary condition.

4.2 Recommendations

Further studies are recommended on application of the developed model for plates with other boundary conditions, varying thickness and plate on elastic foundation.

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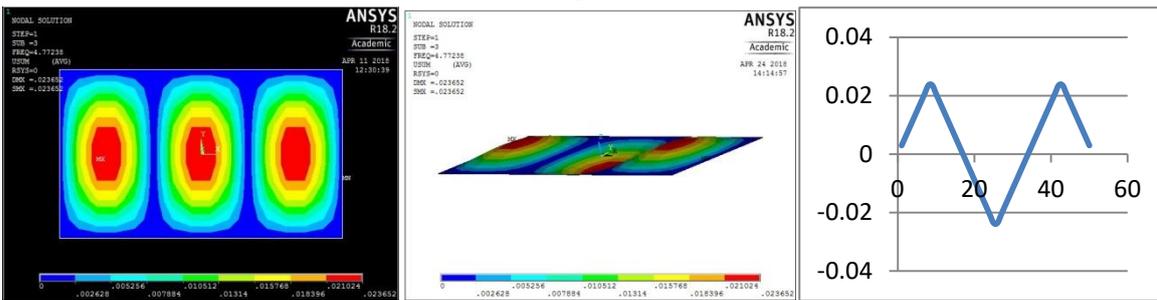
APPENDIX

Plots of Some Representative Mode Shapes
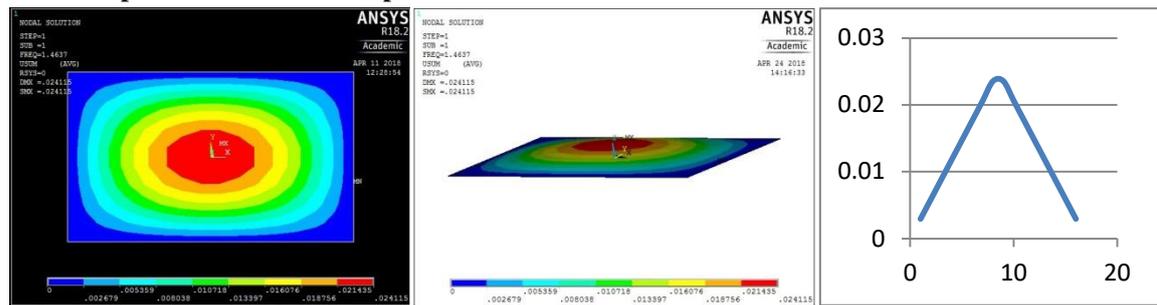
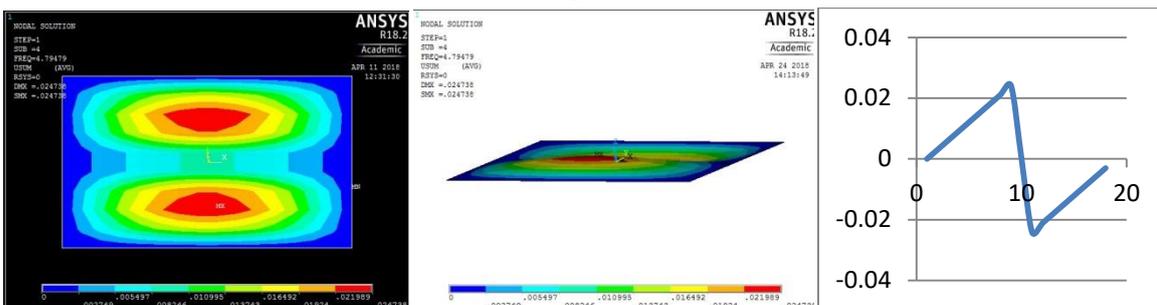1st vibration mode ω_1 = 9.262356191rad/sec
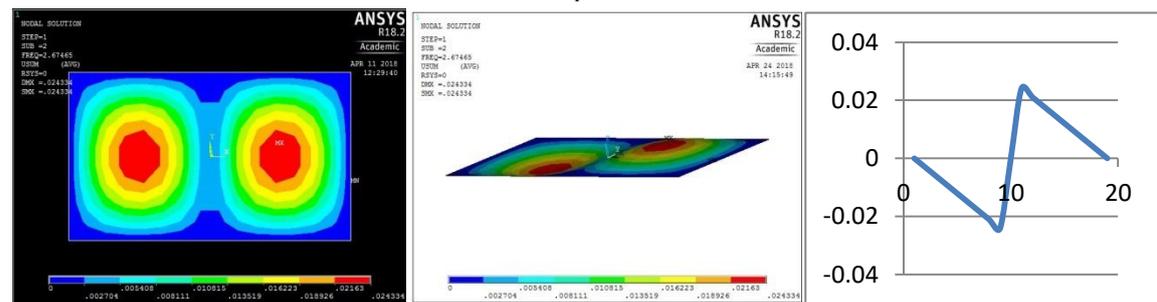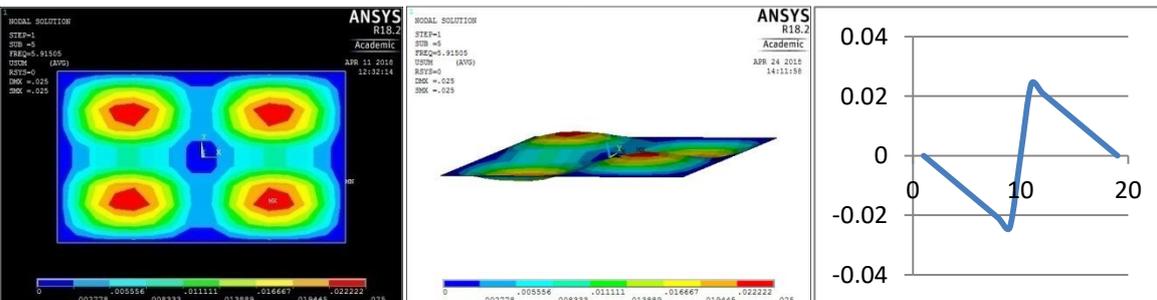

2nd vibration mode ω_2 = 16.80500742 rad/sec

3rd vibration mode ω_3 = 29.98587356 rad/sec

4th vibration mode ω_4 = 30.12661691 rad/sec

5th vibration mode ω_5 = 37.16566941 rad/sec

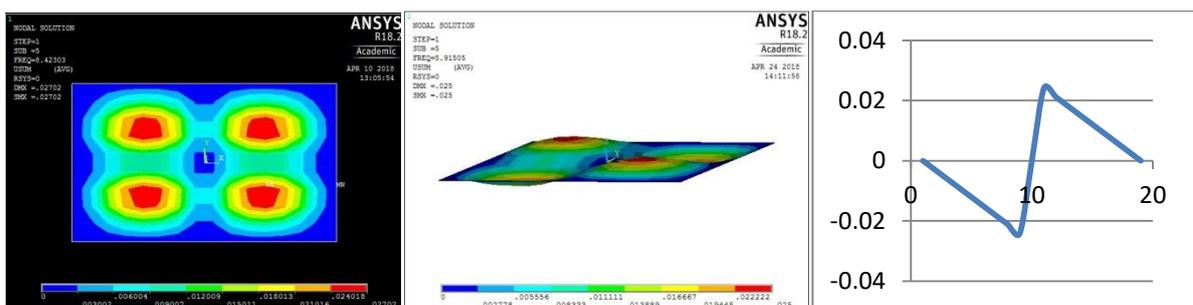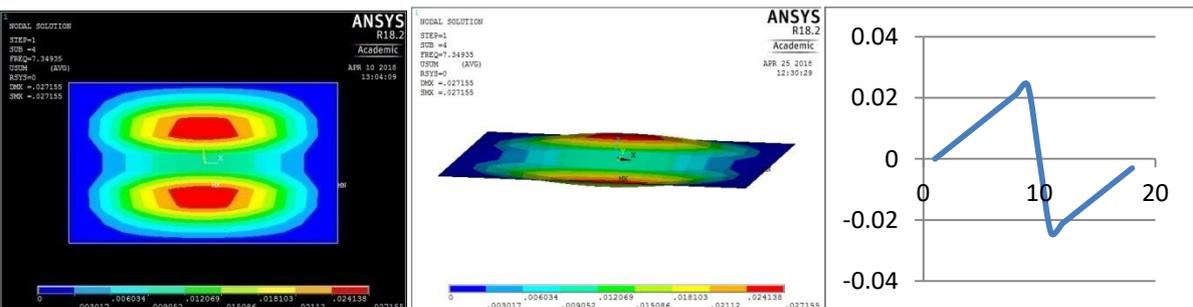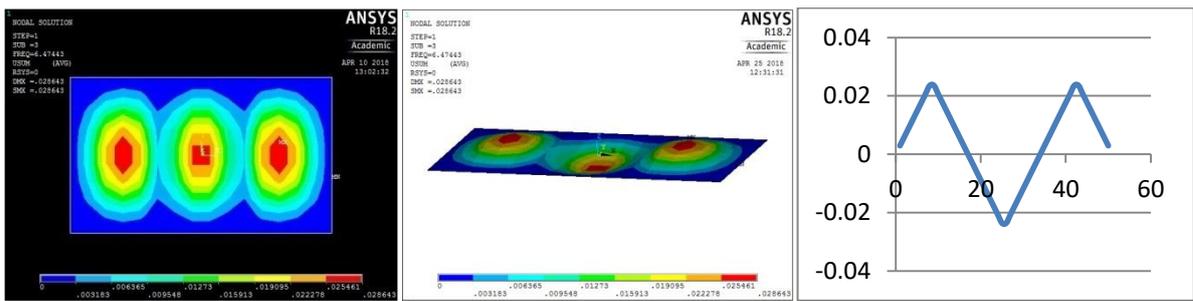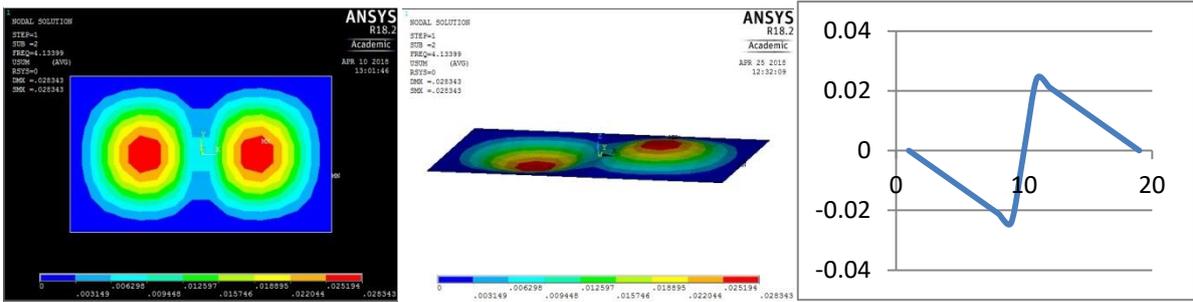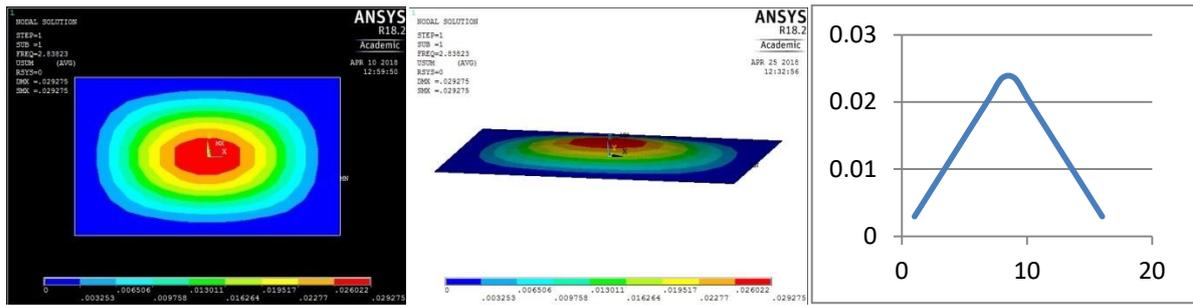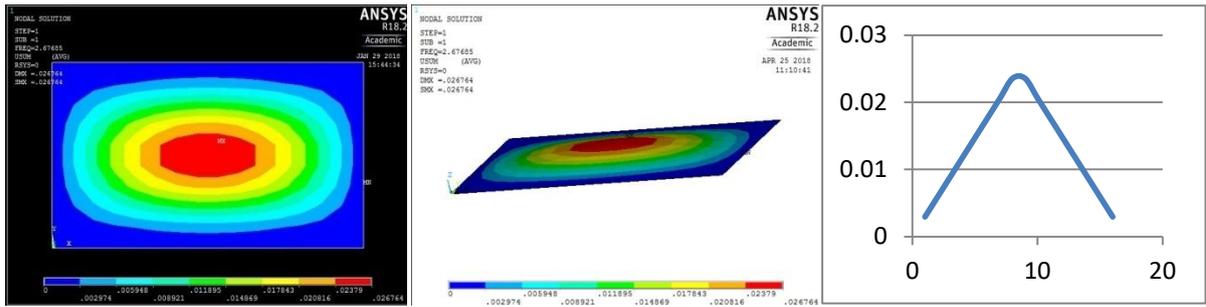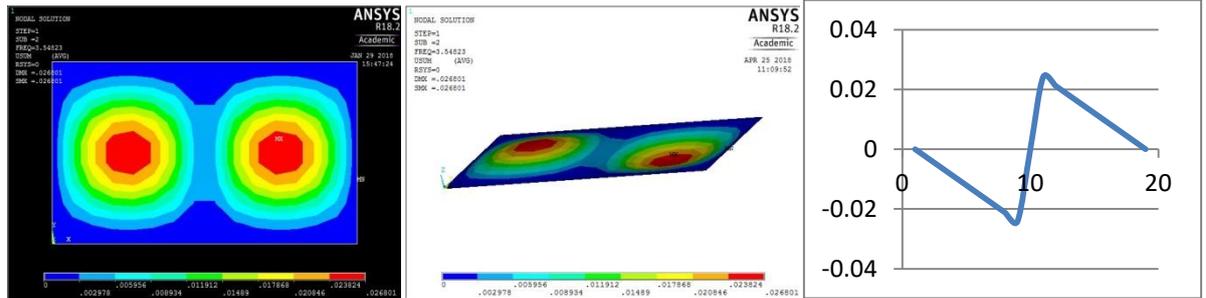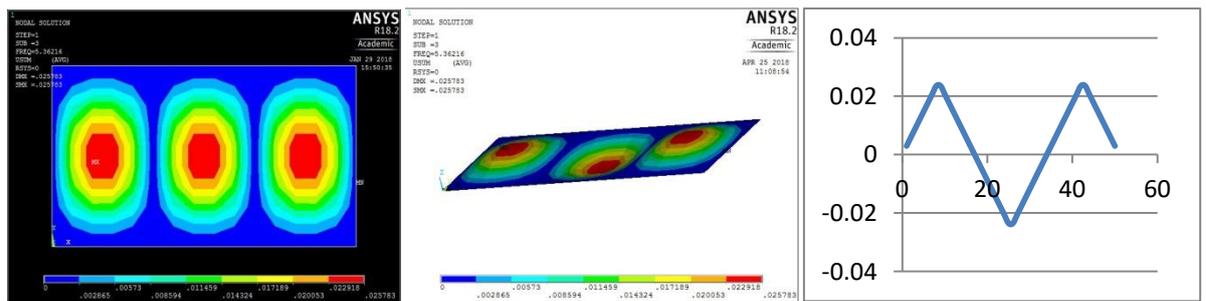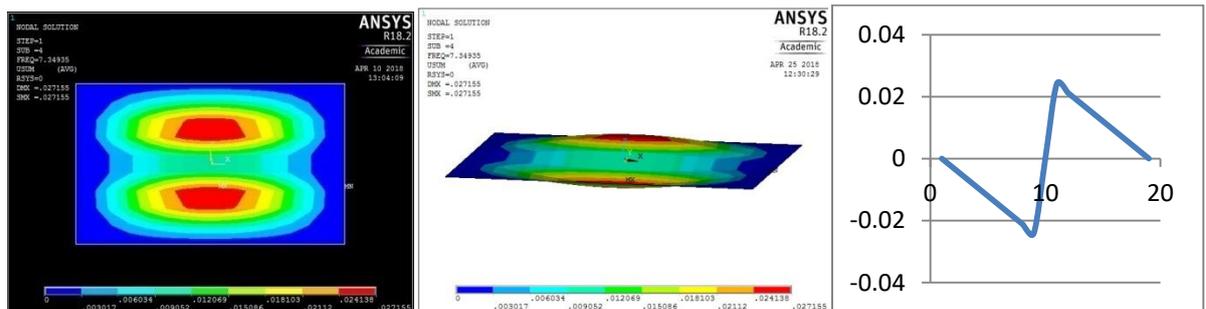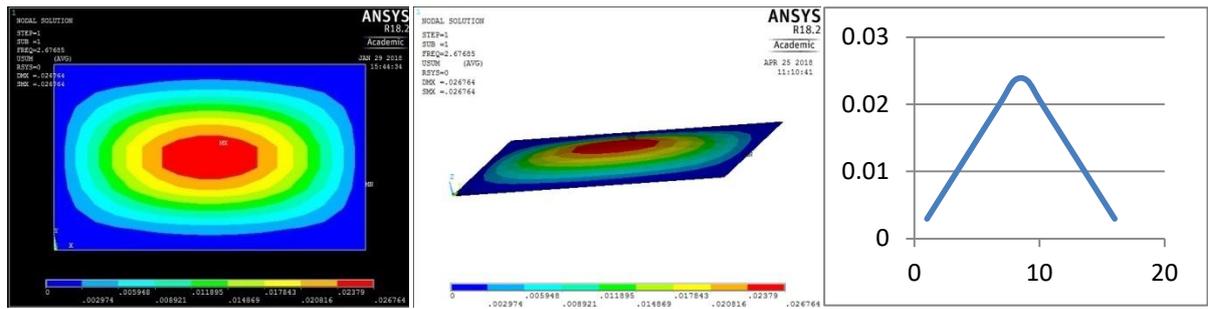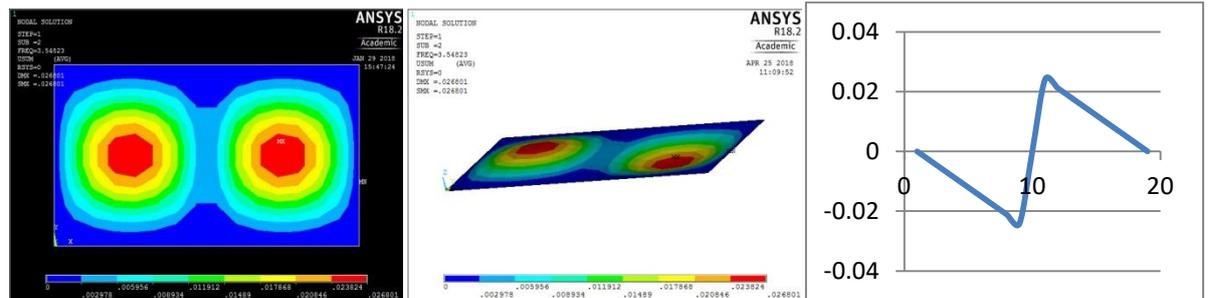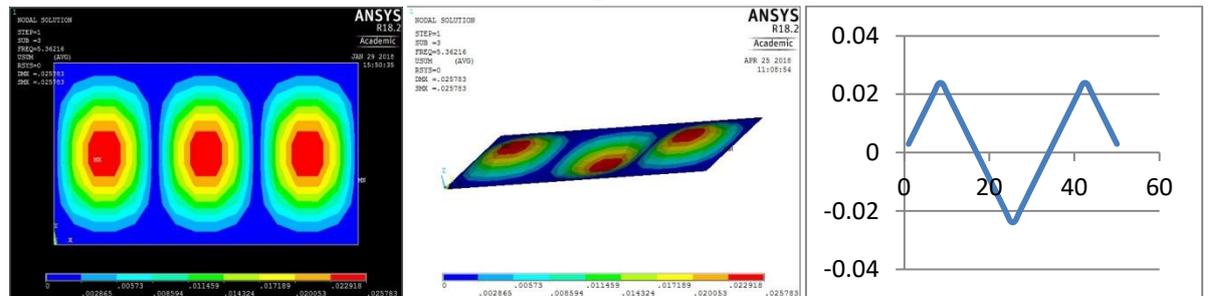**Figure A1: The first five mode shapes of SSSS plate (aspect ratio = 0.6)**

1st vibration mode $\omega_1 = 17.82849370$ rad/sec

2nd vibration mode $\omega_2 = 25.97468806$ rad/sec

3rd vibration mode $\omega_3 = 40.67985495$ rad/sec

4th vibration mode $\omega_4 = 46.1776421$ rad/sec

5th vibration mode $\omega_5 = 52.92326984$ rad/sec

Figure A2: The first five mode shapes of CCCC plate (aspect ratio = 0.6)

1st vibration mode ω_1 = 16.763915 rad/sec

2nd vibration mode ω_2 = 22.29399811 rad/sec

3rd vibration mode ω_3 = 33.69169625 rad/sec

4th vibration mode ω_4 = 45.66493418 rad/sec
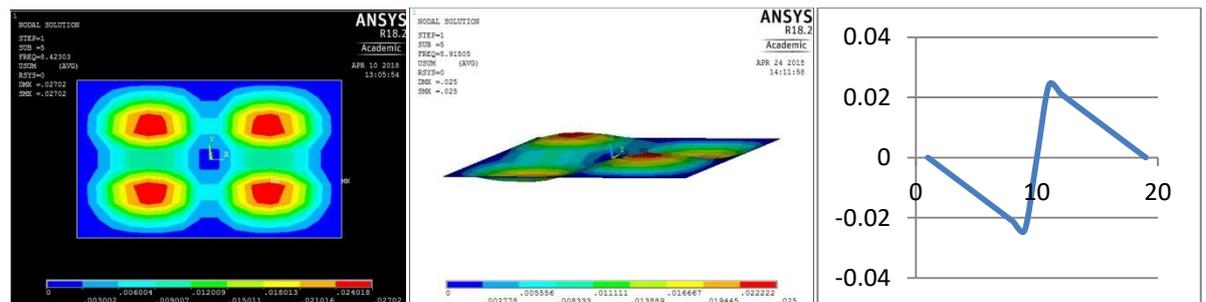
5th vibration mode ω_5 = 50.87809302 rad/sec

Figure A3: The first five mode shapes of CSCS plate (aspect ratio = 0.6)
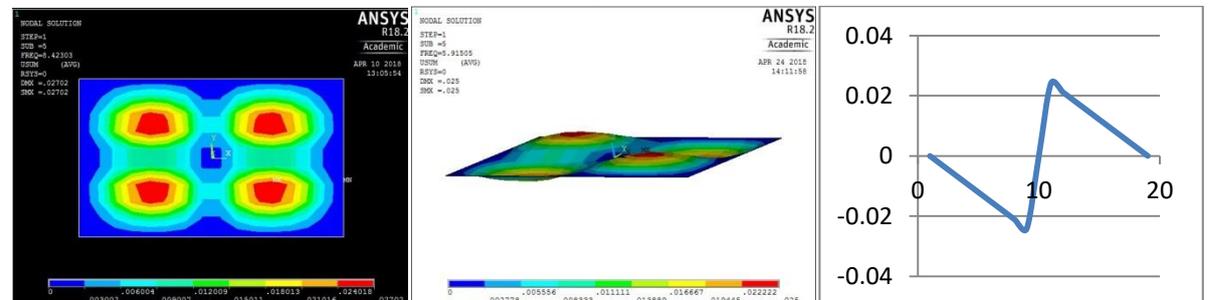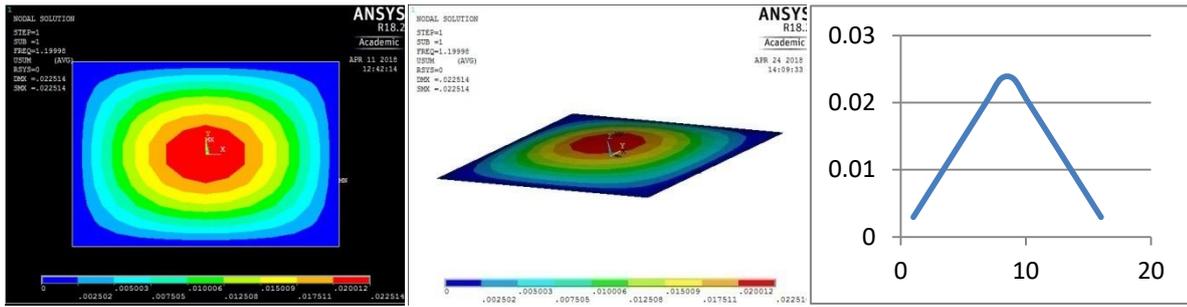
1st vibration mode $\omega_1 = 13.034195$ rad/sec

2nd vibration mode $\omega_2 = 21.01348494$ rad/sec

3rd vibration mode $\omega_3 = 35.04697932$ rad/sec
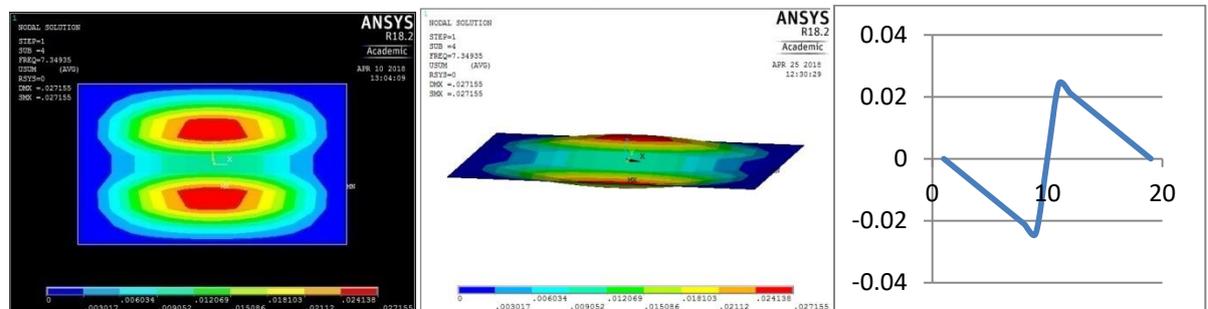
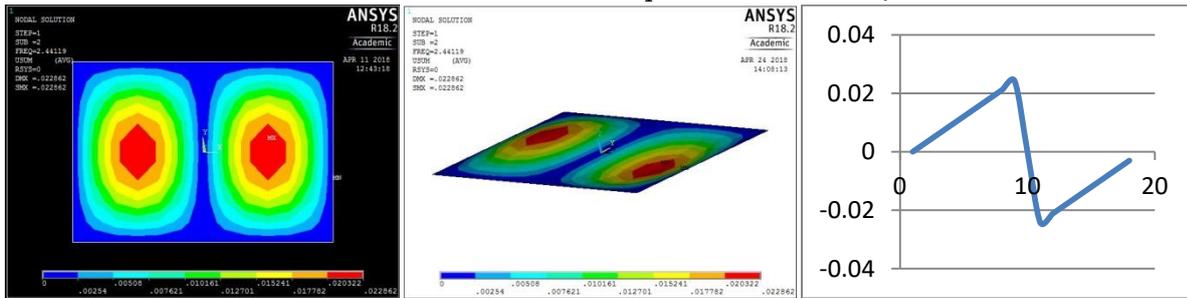4th vibration mode $\omega_4 = 36.66078441$ rad/sec
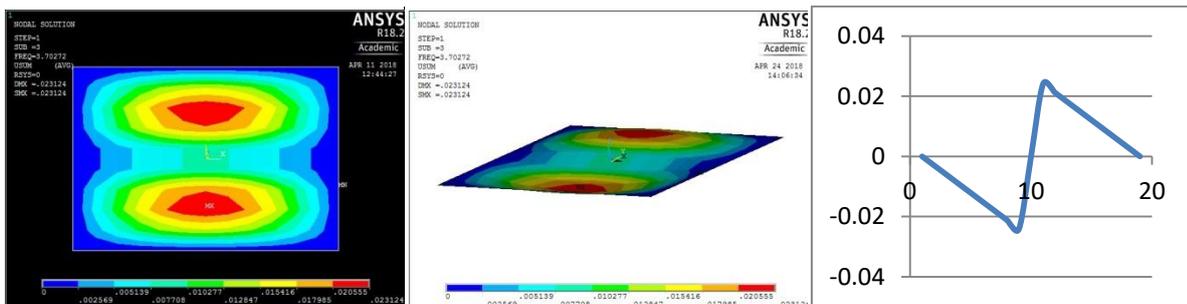

5th vibration mode $\omega_5 = 44.57291657$ rad/sec

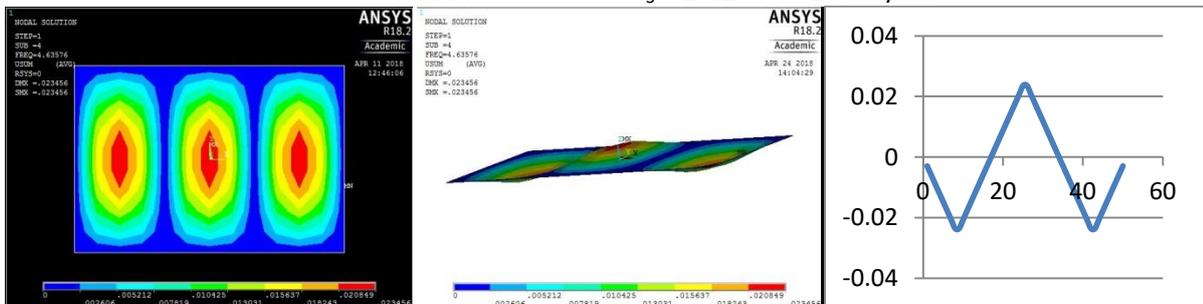**Figure A4: The first five mode shapes of CCSS plate (aspect ratio = 0.6)**

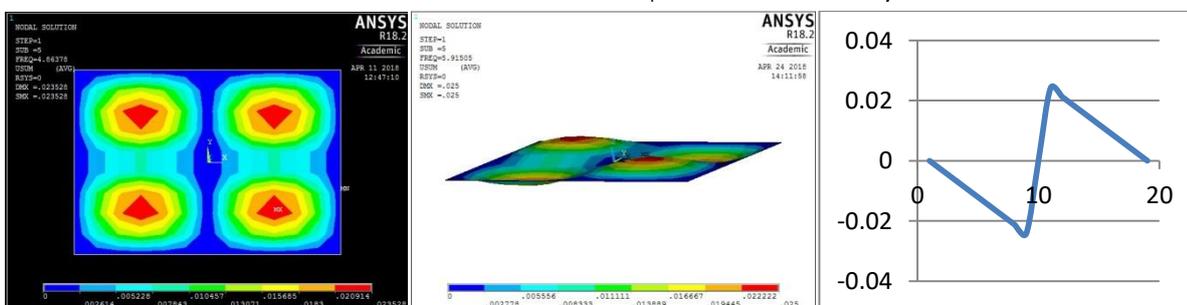1st vibration mode ω_1 = 7.596269446 rad/sec

2nd vibration mode ω_2 = 15.33851197 rad/sec

3rd vibration mode ω_3 = 23.26475024 rad/sec

4th vibration mode ω_4 = 29.12759045 rad/sec

5th vibration mode ω_5 = 30.5601567 rad/sec

Figure A5: The First Five Mode Shapes of SSSS Plate (aspect ratio = 0.7)